Half-range solutions of Sturm-Liouville kinetic equations

1986 J. Phys. A: Math. Gen. 19 1485

(http://iopscience.iop.org/0305-4470/19/8/029)

J. Phys. A: Math. Gen. 19 (1986) 1485-1493. Printed in Great Britain

Half-range solutions of Sturm–Liouville kinetic equations

P Diţă
Institute of Physics and Nuclear Engineering, PO Box MG6, Bucharest, Romania

Received 7 June 1985

Abstract. A simple method for solving boundary value problems for kinetic equations is proposed which by separation of variables leads to indefinite Sturm–Liouville operators. It is shown that the boundary conditions allow the same treatment of the problems as that for definite operators.

1. Introduction

The purpose of this paper is to solve the stationary Sturm–Liouville equation

$$\begin{gathered} r(x)\frac{\partial f(x,y)}{\partial y}=\frac{\partial}{\partial x}\left(p(x)\frac{\partial f(x,y)}{\partial x}\right)-q(x)f(x,y) \\ y\in[0,\infty) \qquad x\in I=(a,b) \qquad -\infty\leqslant a<0, 0<b\leqslant\infty \end{gathered} \tag{1.1}$$

subject, alternatively, to the following boundary conditions:

$$\text{(a)} \qquad \begin{aligned} &f(x,0)=\varphi(x) && \text{if } r(x)>0 \\ &f(x,y)=\mathrm{O}(1) \text{ or } \mathrm{o}(1) && \text{as } y\to\infty \end{aligned} \tag{1.2a}$$

or, with a completely absorbing boundary at $y=0$

$$\text{(b)} \qquad \begin{aligned} &f(x,0)=0 && \text{if } r(x)>0 \\ &f(x,y)\to h(x,y)=yg_1(x)-g_2(x) && \text{as } y\to\infty \end{aligned} \tag{1.2b}$$

where $h(x,y)$ is a diffusion solution of equation (1.1).

The salient feature of the problem is that $r(x)$ changes sign on I. In this paper we shall suppose that $r(x)$ has only one sign change on I, an assumption which seems to be sufficient for treating concrete physical problems and, without loss of generality, we may write $r(x)=xs(x)$, with $s(x)$ a positive function on I.

Boundary value problems of this type arise as various kinetic equations. The Fokker–Planck equation (Wang and Uhlenbeck 1945, Pagani 1970, Beals and Protopopescu 1983, Diţă 1985) has the form

$$\begin{gathered} v\frac{\partial f(x,v)}{\partial x}=\zeta\frac{\partial}{\partial v}\left(v+\alpha\frac{\partial}{\partial v}\right)f(x,v) \\ x\in[0,\infty) \qquad v\in I=(-\infty,\infty). \end{gathered} \tag{1.3}$$

Here ζ is the friction coefficient and $\alpha=kT/m$. If we make the transformation

$$f(x,v)=\exp(-t^2/2)g(x,t) \qquad t=(2\alpha)^{-1/2}v$$

0305-4470/86/081485+09\$02.50

equation (1.3) takes the form

$$2t\zeta^{-1}(2\alpha)^{1/2}[\partial g(x, t)]/[\partial x] = (\partial^2/\partial t^2 + 1 - t^2)g(x, t). \tag{1.4}$$

Bothe's model for electron scattering (Bothe 1929, Beals 1977) leads to the equation

$$\mu \frac{\partial f(x, \mu)}{\partial x} = \frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial f(x, \mu)}{\partial \mu}\right)$$

$$x \in [0, \infty) \qquad \mu \in I = (-1, 1)$$

where $f(x, \mu)$ has finite limits as $\mu \to \pm 1$.

The scattering of plasma waves by random density fluctuations (Fisch and Kruskal 1980) is described by the equation

$$\sin\theta \frac{\partial f(x, \theta)}{\partial x} = \frac{\partial^2 f}{\partial \theta^2}$$

$$x \in [0, \infty) \qquad \theta \in I = (-\pi, \pi), \text{etc.}$$

Although some of them are quite old, the above problems proved to be difficult to solve.

The Fokker-Planck equation (1.3) with the boundary condition $f(0, v) = \alpha f(0, -v)$, $0 \leqslant \alpha \leqslant 1$ at $x = 0$ and exponential decay as $x \to \infty$ was solved by Beals and Protopopescu (1983). An analytic solution for the same equation but with a completely absorbing boundary at $x = 0$ and asymptotic behaviour $f(x, v) \to (x - v\zeta^{-1}) \exp(-v^2/2\alpha)$ as $x \to \infty$ was given recently by Diţă (1985).

In a recent preprint Klaus *et al* (1985) have developed methods of constructing explicit representations for the solutions of the general problem (1.1)-(1.2a). Their methods allow us to compute, in principle, the so-called albedo operator, in terms of which are found the solutions.

The object of this paper is to present an alternative simpler method for solving the problems (1.1)-(1.2), based on an eigenfunction expansion. Our approach to the general equation (1.1) is the use of the separation of variables. The main fact is that, although formally the resulting eigenvalue equation is not of the usual form, it can be treated by the usual methods, since the boundary conditions (1.2a)-(1.2b) allow it.

The paper is organised as follows: in § 2 we solve the problem (1.1)-(1.2a) by finding a complete set of elementary solutions of equation (1.1) on the half-range interval where $r(x) > 0$. In § 3 we treat the same equation but with a completely absorbing boundary at $y = 0$. The next section contains two examples, and the paper ends with a conclusion.

2. A complete set of elementary solutions

We treat equation (1.1) by separation of variables

$$f(x, y) = \exp(-\lambda y)h(x)$$

which leads to the eigenvalue problem

$$\begin{aligned} &-(\mathrm{d}/\mathrm{d}x)(p(x)\,\mathrm{d}h/\mathrm{d}x) + q(x)h = \lambda r(x)h \\ &x \in I = (a, b) \qquad -\infty \leqslant a < 0, 0 < b \leqslant \infty. \end{aligned} \tag{2.1}$$

If we consider the above Sturm-Liouville operator on the interval I, we obtain an indefinite problem since $r(x)$ changes sign there. This is the common method that was used by almost all those who studied equations of this form (Baouendi and Grisvard 1968, Pagani 1969, Beals 1977, Beals and Protopopescu 1983).

Our method consists in looking at the operator (2.1) where the boundary conditions are given, i.e. where $r(x)>0$. In the following we show that this is possible since the boundary conditions (1.2a) and (1.2b) do not impose any boundary condition upon the operator (2.1). In other words, we are free to choose at will the boundary conditions which make the differential expression (2.1) a self-adjoint operator.

For simplicity, we shall suppose that $p(x)$, $q(x)$ and $r(x)$ are continuous functions over any finite interval contained in I. The singularities of the operator (2.1) arise from the singularities of $p(x)$, $q(x)$ and $r(x)$ at one end or both ends of the interval, or from the extension of I to infinity in one direction or both.

If we want a functional calculus, i.e. spectral decompositions, expansion theorems, etc, we have to give a sense to the differential expression (2.1). We can do that by providing equation (2.1) with appropriate boundary conditions. In general, these conditions are provided by the boundary conditions to which the solution $f(x, y)$ is subject. But it is easily seen that the relations (1.2a) and (1.2b) do not impose any definite boundary condition upon $h(x)$. However, they suggest we look at the differential operator (2.1) on the half-range interval $J=[0, b)$. On this interval $r(x)>0$, the Sturm-Liouville problem is definite and the spectral theory is done in the Hilbert space $L_2(J, r(x)\,\mathrm{d}x)$.

The differential operator (2.1) has $x=0$ as a regular point, i.e. both the independent solutions of (2.1) are local $L_2(J, r(x)\,\mathrm{d}x)$, from which we conclude that its deficiency indices are either (1, 1) or (2, 2). For terminology see Reed and Simon (1975).

In the first case the operator (2.1) becomes self-adjoint by imposing one boundary condition at the non-singular end $x=0$, whose general form is

$$h(0)\cos\alpha - p(x)\frac{\mathrm{d}h}{\mathrm{d}x}\bigg|_{x=0}\sin\alpha = 0 \qquad \alpha\in[0, 2\pi). \tag{2.2}$$

In the second case, we have to supplement (2.2) by a boundary condition at the other end $x=b$

$$h(b)\cos\beta - p(x)\frac{\mathrm{d}h}{\mathrm{d}x}\bigg|_{x=b}\sin\beta = 0 \qquad \beta\in[0, 2\pi). \tag{2.3}$$

By imposing the boundary condition (2.2) (or (2.2) and (2.3), respectively) we get for each α (α and β) a self-adjoint extension, each extension providing us with a complete set of eigenfunctions.

The problems (1.1)-(1.2) can be solved in full generality only when there are self-adjoint extensions of the operator (2.1) which have a non-negative spectrum.

This condition is equivalent to the positivity of Friedrich's extension. The restriction arises from the 'physical' condition that the solution $f(x, y)$ should not grow exponentially as $y\to\infty$.

The problem (2.1)-(2.3) being well defined, we get by standard methods (Titchmarsh 1962) complete orthonormal systems of eigenfunctions, which we write in the form $(u_n(x))_{n=0}^{\infty}$, although the operator may have a continuous spectrum.

The functions $(\exp(-\lambda_n y)u_n(x))_{n=0}^{\infty}$ are elementary solutions of equation (1.1) and, consequently, a general solution has the form

$$f(x, y) = \sum_{n=0}^{\infty} c_n \exp(-\lambda_n y)u_n(x) \tag{2.4}$$

where c_n are arbitrary constants. These coefficients are found from the boundary condition at $y=0$

$$f(x,0)=\varphi(x)=\sum_{n=0}^{\infty} c_n u_n(x)$$

from which we get

$$c_n=\int_0^b \varphi(x)u_n(x)r(x)\,\mathrm{d}x \qquad n=0,1,\ldots.$$

Thus the unique solution of the problem (1.1) and (1.2a) can be written as

$$f(x,y)=\sum_{n=0}^{\infty}\exp(-\lambda_n y)u_n(x)\int_0^b \varphi(t)u_n(t)r(t)\,\mathrm{d}t. \tag{2.5}$$

It is easily seen from (2.5) that if $\lambda=0$ is in the spectrum of the self-adjoint extension, $f(x,y)=\mathrm{O}(1)$ as $y\to\infty$, while in other cases the solution is $\mathrm{o}(1)$ as $y\to\infty$.

3. Completely absorbing boundary

The boundary value problem (1.1)-(1.2b) is a little more difficult, and in the case where the deficiency indices are (2, 2) the boundary conditions (1.2b) are not sufficient to determine a unique solution. By adding supplementary information, such as the asymptotic behaviour of the particle density and/or the particle flux, we can again obtain a unique solution.

If we try to use the same procedure as in the preceding section, since now $\varphi(x)=0$, we arrive at $c_n=0$, i.e. to the null solution. Of course the null solution satisfies the boundary condition at $y=0$, but does not verify the boundary condition at $y=\infty$.

We can bypass this difficulty by adding to the linear independent set of functions $(\exp(-\lambda_n y)u_n(x))_{n=0}^{\infty}$ one or more functions which are solutions of equation (1.1). In this way we obtain a linear dependent set of vectors and now we can write an expansion like (2.4), where not all the coefficients are identically zero. As a general rule we shall add only one such function, obtaining in a certain sense a minimal solution. If we add more functions we have to provide supplementary boundary conditions in order to find unique solutions. Thus we have to find new solutions of equation (1.1) which are not of the separated variables form.

The practitioners in the kinetic equations field know that these equations have another type of solution, the so-called diffusion solution, whose general form was conjectured by Fisch and Kruskal (1980) to be $h(x,y)=y-g(x)$. In fact equation (1.1) has a solution of the form

$$h(x,y)=yg_1(x)-g_2(x) \tag{3.1}$$

where g_1 and g_2 satisfy the following coupled equations

$$L_x g_1(x)=0 \tag{3.2a}$$

$$L_x g_2(x)=r(x)g_1(x) \tag{3.2b}$$

where L_x denotes the operator

$$L_x=-(\mathrm{d}/\mathrm{d}x)p(x)(\mathrm{d}/\mathrm{d}x)+q(x).$$

L_x is a second order differential operator and the most general solution $h(x, y)$ depends on four arbitrary constants. Thus the minimal extension of the set $(\exp(-\lambda_n y)u_n(x))_{n=0}^{\infty}$, obtained by adding only one such solution, requires four new constraints in order to find a unique solution.

However one parameter is redundant and can be taken as zero. An argument proving this is as follows.

In the preceding section we found that the asymptotic behaviour of the solution requires the non-negativity of the spectrum of the corresponding self-adjoint extension. If such an extension exists, then, in general, there exists another extension, which may coincide with the previous one, which has $\lambda = 0$ in its spectrum. (If the so-called soft extension is strictly positive, $\lambda = 0$ is not in the spectrum of any extension.)

We consider that extension which has $\lambda = 0$ in its spectrum and let $(u_n(x))_{n=0}^{\infty}$ be its eigenfunctions.

The solution of equation (3.2*b*) is determined up to the general solution of the homogeneous equation $L_x g = 0$, which is the eigenvalue equation for $\lambda = 0$. Thus one of the two independent solutions of $L_x g = 0$ satisfies the boundary conditions of the corresponding extension, and coincides with $u_0(x)$. Hence we can set to zero the coefficient of this solution appearing in $h(x, y)$, since the solution of equation (1.1) has the form

$$f(x, y) = C\left(yg_1(x) - g_2(x) + \sum_{n=0}^{\infty} c_n \exp(-\lambda_n y)u_n(x)\right). \tag{3.3}$$

Here we have chosen one of the remaining parameters as a multiplicative constant.

In equation (3.3), $g_2(x)$ consists of a particular solution of the inhomogeneous equation (3.2*b*), plus the solution, multiplied by an arbitrary constant, of $L_x g = 0$ that does not satisfy the boundary condition of the self-adjoint extension. Thus $g_2(x)$ depends only upon three parameters, two of them being introduced by the general solution $g_1(x)$ of equation (3.2*a*).

If the deficiency indices are (1, 1) and the spectrum is discrete, one of the solutions of the homogeneous equations (3.2*a*, *b*) is not in the Hilbert space $L_2(J, r(x)\,\mathrm{d}x)$ and we have to reject it. In this case $g_1(x)$ and $g_2(x)$ depend upon one parameter which we have chosen in (3.3) as a multiplicative constant. It can be found from the asymptotic behaviour of $f(x, y)$. Indeed the second boundary condition (1.2*b*) requires

$$f(x, y) \to yg_1(x) - g_2(x) \qquad \text{as } y \to \infty$$

and we obtain $C = 1$.

The coefficients c_n are determined by the boundary condition at $y = 0$:

$$0 = -g_2(x) + \sum_{n=0}^{\infty} c_n u_n(x)$$

from which we obtain

$$c_n = \int_0^b g_2(x)u_n(x)r(x)\,\mathrm{d}x \qquad n = 0, 1, \ldots.$$

We find that

$$f(x, y) = yg_1(x) - g_2(x) + \sum_{n=0}^{\infty} \exp(-\lambda_n y)u_n(x) \int_0^b g_2(t)u_n(t)r(t)\,\mathrm{d}t \tag{3.4}$$

is the solution of the problem (1.1)–(1.2*b*).

If the spectrum is continuous and coincides with the positive semi-axis $[0, \infty)$ the above arguments may not be true as the following simple example shows:

$$x\frac{\partial f(x,y)}{\partial y}=\frac{\partial^2 f(x,y)}{\partial x^2}$$

$$y\in[0,\infty) \qquad x\in I=(-\infty,\infty).$$

This equation has the diffusion solution

$$h(x,y)=y(12ax+6b)+ax^4+bx^3+cx+d$$

where a, b, c, $d \in C$ are arbitrary constants. Here both solutions of $L_x g=0$ are not in $L_2(R_+, x\,\mathrm{d}x)$.

If the spectrum is continuous but $\lambda=0$ is an isolated eigenvalue, the above arguments are again true.

If the deficiency indices are $(2, 2)$, the solution (3.4) still contains two arbitrary constants, both in the diffusion solution $h(x, y)$. Their origin is in the fact that the equation $L_x g=0$ has two independent solutions that are both in $L_2(J, r(x)\,\mathrm{d}x)$.

These parameters can be fixed, specifying, for example, the asymptotic behaviour of the particle density and/or the particle flux.

4. Examples

In this section we treat two examples, the first being the Fokker–Planck equation (1.3).

We have shown that by the transformation

$$f(x,v)=\exp(-t^2/2)g(x,t) \qquad t=(2\alpha)^{-1/2}v$$

the Fokker–Planck equation takes the form (1.4). If we make the ansatz

$$g(x,t)=\exp(-\lambda_1 x)h(t)$$

where $\lambda_1=(2\alpha)^{-1/2}\zeta\lambda$ we obtain the equation

$$(-\mathrm{d}^2/\mathrm{d}t^2+t^2-1)h=2t\lambda h. \tag{4.1}$$

Two independent solutions of equation (4.1) are $y_1=D_{\lambda^2/2}(\sqrt{2}(t-\lambda))$ and $y_2=D_{-\lambda^2/2-1}(\mathrm{i}\sqrt{2}(t-\lambda))$, where $D_\nu(z)$ denotes the parabolic cylinder function.

In this case $I=(-\infty,\infty)$ and $J=[0,\infty)$, and we look for a solution $f(x, v)$ such that

$$f(0,v)=\varphi(v) \qquad \text{for } v\in J.$$

If we make the change of variable $t=\sqrt{s}$ on J, equation (4.1) has the form

$$\begin{aligned}&-(\mathrm{d}/\mathrm{d}s)[2\sqrt{s}\,(\mathrm{d}y(s)/\mathrm{d}s)]+\tfrac{1}{2}(\sqrt{s}-1/\sqrt{s})y(s)=\lambda y(s)\\ &y(s)=h(t(s)) \qquad s\in J\end{aligned} \tag{4.2}$$

which is a standard Sturm–Liouville equation. The deficiency indices of the operator (4.2) are $(1, 1)$, since $y_2 \notin L_2(R_+, \mathrm{d}s)$.

We choose $\alpha=0$ in equation (2.2) which leads to the simplest form for the eigenvalue equation. The eigenvalues are the roots of the equation

$$D_{\lambda^2/2}(-\lambda\sqrt{2})=0$$

are positive and satisfy $(2n)^{1/2} < \lambda_n < (2n+2)^{1/2}$, $n = 0, 1, \ldots$. The orthonormal eigenfunctions are

$$u_n(s) = 2^{-1/4} D_{\lambda_n^2/2}(\sqrt{2}(\sqrt{s} - \lambda_n)) / D'_{\lambda_n^2/2}(-\lambda_n\sqrt{2})$$

where $D'_\nu(z) = (\mathrm{d}/\mathrm{d}z) D_\nu(z)$.

The solution of the boundary value problem (1.2*a*)-(1.3) is

$$f(x, v) = 2^{-1/2}\alpha^{-1} \sum_{n=0}^{\infty} \exp\left(-\frac{v^2}{4\alpha} - \frac{\zeta\lambda_n x}{\sqrt{2\alpha}}\right) \frac{D_{\lambda_n^2/2}(\alpha^{-1/2}v - \lambda_n\sqrt{2})}{[D'_{\lambda_n^2/2}(-\lambda_n\sqrt{2})]^2}$$
$$\times \int_0^\infty t \exp(t^2/4\alpha) D_{\lambda_n^2/2}(\alpha^{-1/2}t - \lambda_n\sqrt{2})\varphi(t)\,\mathrm{d}t. \tag{4.3}$$

As concerns the second problem, that with a completely absorbing boundary at $x = 0$, we know that equation (1.3) has a diffusion solution (Pagani 1970) of the form

$$h(x, v) = (x - v\zeta^{-1}) \exp(-v^2/2\alpha).$$

Hence

$$f(x, v) = h(x, v) + 2^{-1/4} \sum_{n=0}^{\infty} d_n \exp\left(-\frac{v^2}{4\alpha} - \frac{\zeta\lambda_n x}{\sqrt{2\alpha}}\right) \frac{D_{\lambda_n^2/2}(\alpha^{-1/2}v - \lambda_n\sqrt{2})}{D'_{\lambda_n^2/2}(-\lambda_n\sqrt{2})}. \tag{4.4}$$

The coefficients d_n are given by the relation (3.4), and in this case they have the form

$$c_n = \zeta^{-1}(2\alpha)^{1/2} \int_0^\infty s^{1/2}\,\mathrm{e}^{-s/2} u_n(s)\,\mathrm{d}s.$$

The solution (4.4) can be written

$$f(x, v) = (x - v\zeta^{-1}) \exp(-v^2/2\alpha) + 2^{-1/2} \sum_{n=0}^{\infty} \exp\left(-\frac{v^2}{4\alpha} - \zeta(2\alpha)^{-1/2}\lambda_n x\right)$$
$$\times \frac{D_{\lambda_n^2/2}(\alpha^{-1/2}v - \lambda_n\sqrt{2})}{[D'_{\lambda_n^2/2}(-\lambda_n\sqrt{2})]^2} \int_0^\infty y^2 \exp(-y^2/4\alpha) D_{\lambda_n^2/2}(\alpha^{-1/2}y - \lambda_n\sqrt{2})\,\mathrm{d}y. \tag{4.5}$$

The relations (4.3) and (4.5) show that the solutions are well defined functions also on the interval $v < 0$. Further details can be found in Diţă (1985).

As a second example we consider the equation

$$\begin{aligned} &\operatorname{sgn} x \frac{\partial f(x, y)}{\partial y} = \frac{\partial^2 f(x, y)}{\partial x^2} \\ &y \in [0, \infty) \qquad x \in I = (-1, 1). \end{aligned} \tag{4.6}$$

By separation of variables we find the equation

$$-h'' = \lambda \operatorname{sgn} x h$$

which on the half-range interval $J = [0, 1)$ can be written as

$$h'' + \lambda h = 0 \qquad x \in J.$$

Two independent solutions are $h_1 = \sin x\sqrt{\lambda}$ and $h_2 = \cos x\sqrt{\lambda}$, which are both in $L_2(J, \mathrm{d}x)$. The deficiency indices are (2, 2).

We choose $\alpha = 0$ in equation (2.2) and β arbitrary in equation (2.3). The eigenvalues are given by the equation

$$[(\sin\sqrt{\lambda})/(\sqrt{\lambda})]\cos\beta - \cos\sqrt{\lambda}\sin\beta = 0.$$

The orthonormal eigenfunctions are

$$u_n(x) = \left(\frac{2(1+\lambda_n\tan^2\beta)}{1+\lambda_n\tan^2\beta+\tan\beta}\right)^{1/2}\sin x\sqrt{\lambda_n} \qquad n = 0, 1, \ldots.$$

The solution of the problem (1.1) and (1.2a) is

$$f(x, y) = 2\sum_{n=0}^{\infty}\frac{1+\lambda_n\tan^2\beta}{1+\lambda_n\tan^2\beta+\tan\beta}\exp(-\lambda_n y)\sin x\sqrt{\lambda_n}\int_0^1 \sin t\sqrt{\lambda_n}\varphi(t)\,\mathrm{d}t.$$

If $\beta = 0$ in equation (2.3), the eigenvalues are $\lambda_n = n\pi$ and the eigenfunctions are $u_n(x) = \sqrt{2}\sin n\pi x$, $n = 1, 2, \ldots$.

Equation (4.6) has the diffusion solution

$$h(x, y) = a[y(x+b) + (x^3/6 + bx^2/2)\operatorname{sgn} x + cx + d]$$

where $a, b, c, d \in C$ are arbitrary constants.

We consider that extension for which $\tan\beta = 1$. Then $\lambda = 0$ is the eigenvalue and its orthonormal eigenfunction is $u_0 = \sqrt{3}x$. In this case we can set $c = 0$ and the diffusion solution depends only upon three parameters. One of them is determined from the condition

$$f(x, y) \to yx \qquad \text{as } y \to \infty$$

and we get $a = 1$. The coefficients b and d are still arbitrary and to obtain a unique solution we need two new constraints upon the function $f(x, y)$, etc.

5. Conclusion

In this paper we have shown that the boundary value problems which by separation of variables lead to indefinite Sturm–Liouville operators can be treated by usual methods, since the boundary conditions allow this.

We have shown also that the completely absorbing boundary condition $f(x, 0) = 0$, where $r(x) > 0$, is not sufficient in determining a unique solution. Even specifying the asymptotic behaviour of the solution does not lead, in all cases, to unique solutions. So much care is needed when solving kinetic equations with such a boundary condition, especially when the problem is treated by approximate numerical methods.

Note added in proof. Solutions (2.5) and (3.4) are valid, strictly speaking, only for $x > 0$. However, the Picard–Lindelöf theorem allows their continuation into the $x < 0$ region. If $p(x)$, $q(x)$ and $r(x)$ are analytic functions, this continuation is automatic, as the Fokker–Planck equation shows. In the second example treated in this paper the use of the Picard–Lindelöf theorem shows that, for $x < 0$, $u_n(x)$ has the form

$$u_n(x) = \left(\frac{2(1+\lambda_n\tan^2\beta)}{1+\lambda_n\tan^2\beta+\tan\beta}\right)^{1/2}\sinh x\sqrt{\lambda_n}.$$

References

Baouendi M S and Grisvard P 1968 *J. Funct. Anal.* **2** 352–7
Beals R 1977 *J. Math. Anal. Appl.* **58** 32–45
Beals R and Protopopescu V 1983 *J. Stat. Phys.* **32** 565–84
Bothe W 1929 *Z. Phys.* **54** 101–78

Diţă P 1985 *J. Phys. A: Math. Gen.* **18** 2685–90
Fisch N J and Kruskal M D 1980 *J. Math. Phys.* **21** 740–50
Klaus M, van der Mee C V M and Protopopescu V 1985 *Boston University preprint*
Pagani C D 1969 *Rend. Istituto Lombardo* **103** 618–53
—— 1970 *Boll. Un. Mat. Ital.* **3** 961–86
Reed M and Simon B 1975 *Methods of Modern Mathematical Physics* vol II (New York: Academic)
Titchmarsh E C 1962 *Eigenfunction Expansions* part one (Oxford: Clarendon) 2nd edn
Wang M C and Uhlenbeck G E 1945 *Rev. Mod. Phys.* **17** 323–42